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ON THE STABILITY OF POINTS OF EQUILIBRIUM BRANCHING

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V. I. VOZLINSKII
(Moscow)

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The problem of stability of equilibrium of branching points of finite-dimensional conservative systems whose potential energy depends on a real parameter is considered.

The investigation of branching point stability is fraught with complications due to degeneration of the potential energy second differential at such points. It is shown here that in most common cases stability or instability of a branching point can be assessed by the properties of the equilibrium curve in its neighborhood. The proof is based on geometrical considerations that are independent of the rank of the potential energy Hessian, and is thus valid for systems that do not satisfy Poincaré's conditions [1] (see Remark 3.1 below). When these conditions are satisfied, the problem reduces to a system with a single degree of freedom, and the basis results of the present work are readily obtained by methods of the Poincaré — Chetaev theory of equilibrium bifurcation [1, 2]. Extension to the case of several parameters is considered.

The results obtained by Rumiantsev's method [3] may be applied to problems of steady motion equilibrium.

Investigation of branching point stability is of interest for obtaining a complete picture of distribution of stable points on the equilibrium curve branches and, also, in connection with the "safety" of branching (see Remark 3.2).

1. We use the notation $X \ni x$ for the n-dimensional configuration space; $A \ni \alpha$ for an interval of the numerical axis; $O(x)$, $O(\alpha)$, and $O(x, \alpha)$ for the neighborhoods of points X, A, and XA, respectively, and $\Pi(x, \alpha)$ for the potential energy.

We say that the property dependent on parameter $\alpha \in A$ is specified on X if property $\beta(x, \alpha)$ is specified on XA, i.e. that set $B \subset XA$ (the determining set), at whose points the property $(\beta(x, \alpha) = 1)$ is satisfied, while outside that set it is not satisfied $(\beta(x, \alpha) = 0)$, is specified.

We call set $B \subset XA$ open with respect to α at point $(x^\circ, \alpha^\circ) \in B$ if for any $\varepsilon > 0$; $\alpha^\circ \in A$ is an inner point of the projection of set $B \cap O^\varepsilon(x^\circ, \alpha^\circ)$ on A (i.e. there exists for every α in a fairly small neighborhood of $O(\alpha^\circ)$ a point x such that $(x, \alpha) \in B \cap O^\varepsilon(x^\circ, \alpha^\circ)$). We call the set B open with respect to α (open with respect to α in region $D \subset XA$), if it is open at all its points (at points in D), and open with respect to α relative to set $M \subset XA$, if set $B \cap M$ is open with respect to α.

As an example we consider two systems determined respectively, by the potential energies $\Pi(x, \alpha) = x^4 - \alpha x^2$ and $\Pi(x, \alpha) = -x^4 - \alpha x^2$ (the one-dimensional case). Equilibrium curves of these systems are shown in Fig. 1, where the solid and the dash lines relate to stable (set B^+) and unstable equilibrium, respectively. In both cases set B^+ is open with respect to α, including at point $(0,0)$ in Fig. 1, a, although at that point the first system is not coarse [6]. Simultaneously set B^+ (Fig. 1, a) is is not open with respect to α relative to the straight line $x = 0$. This is close (but not equivalent) to the meaning given in [7] to the noncoarseness of the property in relation to perturbations of parameter α. Thus in these two examples the sets that are open with respect to α correspond to stability. It is shown below that with certain limitations this conclusion is generally valid.

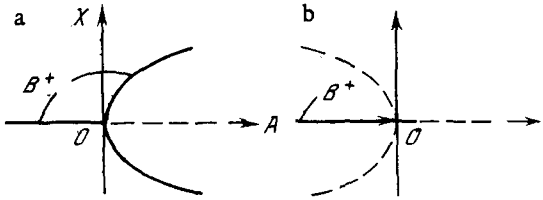

Fig. 1

By analogy to the concept of the equilibrium branching point we say that (x°, α°) is a branching point of the property $\beta(x, \alpha)$ if it is a branching point of solutions of the equation $\beta(x, \alpha) = 1$, i.e.

$$(\forall \varepsilon > 0)\ (\exists x_1,\ x_2,\ \alpha')\ (x_1,\ \alpha'),\ (x_2,\ \alpha') \in \mathrm{B} \cap O^\varepsilon (x^\circ,\ \alpha^\circ) \qquad (1.1)$$

If (1.1) is satisfied for $\alpha' = \alpha^\circ$ we call the branching trivial.

If at point (x°, α°) branching is nontrivial and there exists a neighborhood $O(x^\circ, \alpha^\circ)$ in which set B is empty when $\alpha < \alpha^\circ$ or $\alpha > \alpha^\circ$, that point is called the Poincaré limit branching point, at which set B is obviously not open with respect to α.

We say that a single half-branch of set B (see Fig. 1, a,) adjoins point (x°, α°) on the left, is set B nonempty in any neighborhood of that point for $\alpha < \alpha^\circ$, and there exists region $O(x^\circ, \alpha^\circ)$ in which for every $\alpha < \alpha^\circ$ the solution of the equation $\beta(x, \alpha) = 1$ is unique (i.e. condition (1.1) is not satisfied for $\alpha < \alpha^\circ$ and set B is not empty). The concept of the right-hand single half-branch (Fig. 1, b) is similarly obtained.

We call (x°, α°) the equilibrium or steady point (with respect to x) if x° is the equilibrium position for $\alpha = \alpha^\circ$. The property of steadiness with respect to x is denoted here by $\Pi'(x, \alpha) = 0$. The determining set $\mathrm{B} \subset XA$ is in that case defined by the equilibrium curve. The equilibrium point (x°, α°) is called isolated when for a fixed $\alpha = \alpha^\circ$ there exists a neighborhood that does not contain any other equilibrium points.

We denote by $\Delta\Pi(x, \alpha) > 0$ the property of positive definiteness of potential energy increments with respect to x (i.e. the property of strict minimum of function $\Pi(x, \alpha)$ with respect to x).

Below the potential energy is assumed to be continuous with respect to (x, α), and some of the statements will be based on the further assumption that the potential energy gradient with respect to x is continuous in (x, α).

2. Lemma 2.1. Let (x°, α°) be a point of equilibrium branching to which adjoins the single half-branch C of the equilibrium curve B. We assume that in the neighborhood of that point $\operatorname{grad}_x \Pi (x, \alpha)$ is continuous with respect to (x, α), that property $\Delta\Pi > 0$ is satisfied on C outside point (x°, α°) and that at point (x°, α°) itself that property is violated. Then for $\alpha = \alpha^\circ$ there exists a connected set of equilibrium positions that contain point x° and not coincident with it (i.e. the equilibrium position x° is not isolated when $\alpha = \alpha^\circ$ and there is no discrete accumulation of equilibrium positions).

Proof. It can be assumed without loss of generality that the zero $(\theta, 0)$ of space XA is a branching point, that the half-branch $x = \theta,\ \alpha < 0$ is the half-branch C, and that $\Pi (\theta, \alpha) \equiv 0$.

Let us consider the sequence $\{\alpha_i\},\ \alpha_i \to 0,\ \alpha_i < 0$. By the definition of the single half-branch there exists a neighborhood $D = O(\theta)$ that is independent of α and does not contain equilibriums other than θ when $\alpha < 0$. We denote by $E(\lambda, \alpha_i)$ the set of potential energy level in X for $\alpha = \alpha_i$: $\Pi (x, \alpha_i) = \lambda$ and by $K(\lambda, \alpha_i)$ a component of that set. Owing to property $\Delta\Pi (\theta, \alpha_i) > 0$ there exists for every α_i in the considered sequence D a region Ω_i where all components of the level of function $\Pi (x, \alpha_i)$ are homeomorphic to hyperspheres, and it follows from $\lambda_1 < \lambda_2$ that $\theta \in \omega_1 \subset \omega_2$, where ω_s is the region bounded by component $K(\lambda_s, \alpha_i)$. We call region Ω_i (we have in mind the greatest region in D that has these properties) the region of regularity of functions $\Pi (x, \alpha_i)$. The volumes of regions Ω_i may tend to zero when $\alpha_i \to 0$, but all these regions are unbounded in D owing to the absence in the latter of steady nonzero points. Hence the upper topological bound $\Omega^* = \mathrm{lt}^* \Omega_i$ of sequence $\{\Omega_i\}$ has at least one point at the boundary of region D, obviously including point θ. Since the lower topological bound of that sequence is not empty (point θ belongs to it), hence by Zoretti's theorem [8] the upper topological bound Ω^* is connected, and by virtue of the above does not degenerate into point θ.

We shall show that the connected set $\Omega^\circ \subset \Omega^*$ $(\theta \in \Omega^\circ)$ of zeros of function $\Pi (x, 0)$ that does not degenerate into point θ does exist.

Since at point $(\theta, 0)$ the property $\Delta\Pi > 0$ is violated, hence one of the following conditions must be satisfied.

1°. Every neighborhood $O(\theta)$ has points x': $\Pi (x', 0) < 0$.

2°. There exists a neighborhood $O(\theta)$ where $\Pi (x, 0) \geqslant 0$, and points x°: $\Pi (x^\circ, 0) = 0$ exist in any neighborhood $O(\theta)$.

Condition 1° implies the branching of property $\Pi = 0$ to the left. In that case $\Omega^\circ = \Omega^*$, which is proved as in [9]. This also applies to case 2° when, as in the first case, the branching of property $\Pi = 0$ is to the left.

We shall now prove the existence of set Ω° in case 2°, if there is no left branching of property $\Pi = 0$. A neighborhood of point θ in which $\Pi (x, \alpha_i) > 0$ when $\alpha_i < 0,\ x \neq \theta$ then exists. It can be shown that such neighborhood coincides with D.

Let us consider sequence $\{x_j\},\ x_j \to \theta,\ \Pi (x_j, 0) = 0$ in D. We have $\Pi (x_j, \alpha_i) = \lambda_{ji} > 0$. Let $K_{ji} = K(\lambda_{ji}, \alpha_i)$ be the component that contains point x_j. The definition of the regularity of region Ω_i implies that either $K_{ji} \subset \Omega_i$ or $K_{ji} \cap \Omega_i = \varnothing$. If there exists a j such that the first of these possibilities occurs for an infinite number i, we obtain the i-sequence of regions $\omega_{ji} \subset \Omega_i$ bounded by components K_{ji}. The upper topological bound of that sequence may be taken as the set Ω°, since $x_j \in \Omega^\circ$ $(x_j \neq \theta)$ and $\Pi (x_j, \alpha_i) \to 0$ when $\alpha_i \to 0$ (point x_j is fixed).

If there is no such j, there exists for every j a sequence $\{j\}$ in which almost the whole i-sequence $\{K_{ji}\}$ does not intersect Ω_i. Consequently the boundaries $\Gamma(\Omega_i)$ of these regions separate points x_j and θ [10]. We denote (for some $j \in \{j\}$)

$$\Omega^j = \operatorname*{lt^*}_{i\to\infty}\Omega_i, \quad S_j = \lim_{i\to\infty}\sup_{x\in l_j} \Pi(x, \alpha_i)$$

where l_j represents segment (θ, x_j). Evidently Ω^j intersects $\Gamma(D)$, and $S_j \geqslant 0$. Since the quantity $\operatorname{grad}_x \Pi(x, \alpha)$ is bounded in the neighborhood $O(\theta, 0)$, there exists a number g: $S_j \leqslant g \| x_j \|$ that is independent of j. Hence $\lim S_j = 0$ when $x_j \to \theta$, and since points x_j and θ are separated by set Ω^j we have $\sup \Pi(x, 0) \leqslant S_j$, $x \in \Omega^j$ and it is possible to set $\Omega^\circ = \mathrm{lt^*}\,\Omega^j$, $j \to \infty$.

It has thus been shown that in all considered cases there exists sequence $\{\Omega_i\}$ whose upper topological bound may be assumed to be the set Ω°, since it is connected and belongs to component $K(0, 0)$. We shall now show that Ω° consists of stationary point of function $\Pi(x, 0)$ and, by the same token, prove the lemma. Let us assume the contrary, i.e. that there exists a nonstationary point $x' \in \Omega^\circ$. Then owing to the continuity of $\operatorname{grad}_x \Pi(x, \alpha)$ there exists for every fairly small $|\alpha_i|$ a gradient tube $G_i \subset X$ of function $\Pi(x, \alpha_i)$ which contains point x' and whose all gradient lines intersect the region of negativeness of function $\Pi(x, \alpha_i)$. Simultaneously there exists sequence $\{\alpha_i\}$, $\alpha_i \to 0$ for which the intersection $G_i \cap \Omega_i$ is not empty (since $x' \subset \Omega^\circ$). But any gradient line that intersects the region of regularity of Ω_i cannot intersect the region of negative functions $\Pi(x, \alpha)$ (it can be assumed that such function has no stationary points in the considered length of the gradient tube. This contradiction proves the lemma.

Lemma 2.2. If condition $\Delta\Pi > 0$ is satisfied at an isolated equilibrium point, and in the neighborhood of that point function $\Pi(x, \alpha)$ is continuous, then for every $\varepsilon > 0$ there exists a $\delta > 0$ such that a component of the level of function $\Pi(x, \alpha)$ homeomorphic to the hypersphere surrounding point θ exists in $O^\varepsilon(\theta)$ for every $\alpha \in O^\delta(0)$.

Proof. The idea of the proof was in essence formulated in [11]. Let Ω be the regularity region of function $\Pi(x, 0)$ in the $O^\varepsilon(\theta)$ neighborhood. We specify three components of level in that region: $K_1 = K(\lambda_1, 0)$, $K_2 = K(\lambda_2, 0)$ and $K^* = K(\lambda^*, 0)$ with $\lambda_1 < \lambda^* < \lambda_2$, and denote

$$S_1(\alpha) = \sup \Pi(x, \alpha), \quad x \in K_1; \quad S_2(\alpha) = \inf \Pi(x, \alpha), \quad x \in K_2$$

Since region ω bounded by components K_1 and K_2 does not contain steady points of function $\Pi(x, 0)$, it is possible to select a δ as small as desired and such that for $\alpha \in O^\delta(0)$ region ω does not contain steady points of function $\Pi(x, \alpha)$, that $S_1(\alpha) < S_2(\alpha)$, and that x^*: $S_1(\alpha) < \Pi(x^*, \alpha) < S_2(\alpha)$ exists in ω. Hence the component K_α of function $\Pi(x, \alpha)$ which passes through point x^* is bounded in ω and, since ω does not contain steady points, it is a separatrix in ω, and is consequently homeomorphic to the hypersphere [10].

Corollary. If $\Delta\Pi(x^\circ, \alpha^\circ) > 0$, then set $\Pi' = 0$ is open in α at point (x°, α°). If, furthermore, the neighborhood $O(x^\circ, \alpha^\circ)$ does not contain a trivial branching of equilibrium, set $\Delta\Pi > 0$ is also open with respect to α at point (x°, α°).

At least one steady point of potential energy obviously exists in the region bounded by the component dealt with in Lemma 2.2, which implies that set $\Pi' = 0$ is open with respect to α. If the considered region does not contain trivial branching of equilibrium, the steady points are isolated and, owing to the boundedness from below of function $\Pi(x, \alpha)$ in $O(x^\circ, \alpha^\circ)$, at least one of these is a point of strict minimum of function $\Pi(x, \alpha)$ with respect to x. This implies that set $\Delta\Pi > 0$ is open.

3. Let us assume that for fixed α the potential energy satisfies in addition to the condition of continuity, satisfies the conditions under which inversion of the Lagrange theorem is valid for isolated positions of equilibrium. We thus assume the fulfilment of conditions for which the following statement is valid [12]: if at isolated equilibrium positions the potential energy has no minimum, that equilibrium position is unstable (statement A).

In an isolated equilibrium position any minimum of potential energy is strict and isolated (an isolated minimum implies the existence of a neighborhood that does not contain any other minima).

Various cases in which inversion of the Lagrange theorem is valid were considered in [12 — 14].

The condition of isolation of the equilibrium position given in Statement A was used by Chetaev [12] primarily for eliminating cases of the kind of Painleve's example (see, e.g., [14]). All examples of invalidity of inversion of the Lagrange theorem known to the author relate to nonisolated equilibrium positions.

Remark 3.1. Statement A is evidently valid under the following constraints imposed by Poincaré on potential energy [1].

1°. At the branching point $\operatorname{rank}[\Pi_{ij}(x^\circ, \alpha^\circ)] = n - 1$ ($[\Pi_{ij}]$ is the matrix of second derivatives of potential energy with respect to coordinates) and one of the principal minors of order $n - 1$ is positive.

2°. Outside of branching points of the equilibrium curve $\det[\Pi_{ij}] \neq 0$.

Under these conditions the problem reduces to a system with one degree of freedom.

From Lemmas 2.1 and 2.2 on assumptions made at the beginning of Sect. 3 we obtain several theorems.

Theorem 3.1. In region $D \subset XA$ that does not contain trivial branching of equilibrium the set of stable equilibrium points is open with respect to α.

This theorem with Statement A taken into account is equivalent to the second part of the corollary of Lemma 2.2. An example illustrating this theorem was considered in Sect. 1 (Fig. 1).

Theorem 3.2. Any limit point of branching is unstable.

At a limit point set $\Pi' = 0$ is obviously closed with respect to α, which contradicts the corollary of Lemma 2.2, if it is assumed that the limit point is stable and is an isolated equilibrium point.

Theorem 3.3. Let in the neighborhood of a nontrivial branching point $\operatorname{grad}_x \Pi(x, \alpha)$ be continuous with respect to (x, α), and let a single half-branch of the equilibrium curve adjoin that point. Then the stability properties of the branching point is the same as along the single half-branch.

In fact, let $(\theta, 0)$ be a branching point with a single half-branch adjoining it from the left. It follows from Theorem 3.1 that when point $(\theta, 0)$ is stable, then for all fairly small $|\alpha|$, $\alpha < 0$, in the neighborhood $O(\theta, 0)$ stable equilibrium points must exists. Hence, if the considered single half-branch is unstable, point $(\theta, 0)$ is also unstable. If, however, that branch is stable, the stability of point $(\theta, 0)$ follows from Lemma 2.1.

Remark 3.2. From Lemma 2.2 we obtain the following statement which in essence was proved in [11] (where the proof of a general character was obtained in connection with the analysis of a particular mechanical system): any equilibrium point is stable at the initial perturbation of parameter α. Stability of equilibrium $x = \theta$ at the initial perturbation of parameter α is understood to imply the fulfilment of the following conditions: for any $\varepsilon > 0$ there exists $\delta > 0$ such that any trajectory $x(x^\circ, \alpha^\circ, t)$, where $x^\circ \in O^\delta(\theta)$ and $\alpha^\circ \in O^\delta(0)$, does not emanate from $O^\varepsilon(\theta)$ (X_1 is the phase space).

In this sense it is possible to say that when (x°, α°) is a point of stability change on some branch of the stability curve and is stable, branching at that point is less dangerous than when it is unstable [4, 5].

Remark 3.3. If stability is understood in the sense described above, it is possible to extend Theorem 3.2 to the simplest case of a nonconservative system, viz. the two-dimensional autonomous dynamic system that depends on parameter

$$x^\cdot = f(x, \alpha), \quad x = (x^1, x^2) \tag{3.1}$$

where function $f(x, \alpha)$ is continuously differentiable with respect to x and continuous with respect to α.

Theorem 3.4. If $(\theta, 0)$ is the limit equilibrium point of system (3.1), it is unstable at the initial perturbation of parameter α.

Let us assume the opposite, namely, that the limit equilibrium point $(\theta, 0)$ is stable in the sense defined above. Then for a reasonably small $\delta > 0$ the motion $x = x(x^\circ, \alpha^\circ, t)$, where $x^\circ \in O^\delta(\theta)$ and $\alpha^\circ \in O^\delta(0)$, is bounded in the neighborhood $O(\theta)$. Hence, when $\alpha = \alpha^\circ$ there exists in $O(\theta)$ a limit cycle and the region bounded by it contains at least one equilibrium point. This contradicts the assumption that $(\theta, 0)$ is a limit point.

4. Let us consider the case of m parameters $\alpha_1, \alpha_2, \ldots, \alpha_m$.

Here α is understood to represent the vector of the related m-dimensional space A. The introduced in Sect. 1 definitions of set B open with respect to α, of the point of branching of property $\beta(x, \alpha)$, and of trivial branching apply in this case without any alteration. The concept of the limit point of branching is extended as follows. We call the branching point (x°, α°) of property $\beta(x, \alpha)$ limit point, if the determining set B of that property is not open with respect to α at the considered point (the fact that set $\Pi' = 0$ is not open at the limit point played an important part in the proof of Theorems 3.2 and 3.4).

Let, for example, set B be determined by equation $\alpha_1^2 + \alpha_2^2 - x^2 = 1$ (a hyperboloid of one sheet in the space (α_1, α_2, x), where $m = 2$ and $n = 1$). The set is open with respect to α at all of its points, except at points of the circle $\alpha_1^2 +$

$\alpha_2{}^2 = 1$, $x = 0$ that are limit branching points.

If at least one cylindriical cross section of set B exists and has (x°, α°) as a limit point [15] in the meaning defined in Sect. 1, then it is obviously a limit point in the new meaning.

A cross section is called cylindrical when it is defined by the equality $\alpha = \alpha(s)$ $(\alpha^\circ = \alpha(s^\circ))$, where $s \in S$ is a real parameter. It is assumed that this equality determines a simple part of a curve in space A (the cross section "directrix"). The set $L^* \subset XS$ defined by the equality $\beta(x, \alpha(s)) = 1$ corresponds to the cylindrical cross section $L \subset B \subset X\mathrm{A}$. If one of parameters α_i is taken as parameter s (as in [15]), then L^* is the projection of L on the subspace (x, α_i).

We say that the cylindrical cross section L contains a single half-branch that adjoins point (x°, α°), if a single half-branch of set L^* adjoins point $(x^\circ, s^\circ) \in XS$ in the sense defined in Sect. 1. In that case Lemma 2.1 applies to cross section L (with the substitution of words "cross section L of equilibrium surface B" for "equilibrium curve B"). From this the follows the theorem analogous to Theorem 3.3.

Theorem 4.1. If in the neighborhood of a nontrivial branching point $\mathrm{grad}_x \Pi(x, \alpha)$ is continuous with respect to (x, α) and there exists at least one cylindrical cross section of the equilibrium surface that contains a single half-branch adjoining the considered point, the stability properties of the branching point are the same as along the single half-branch (irrespective of the form of other cross sections).

Lemma 2.2 and Theorems 3.1, 3.2, and 3.4 apply to the case of m parameters without any alteration. The proof is in essence the same as in Sects. 2 and 3.

5. Example 1. Let us consider the example of a pendulum whose horizontal axis rotates around the vertical examined in [3]. Curves of steady motions obtained in [3] for the cases of $0 < \alpha < 1$ (a), $-1/3 < \alpha < 0$ (b), $\alpha < -1/3$ (c) are shown in Fig. 2, where $\beta \geqslant 0$ is a quantity proportional to the square of the generalized momentum that corresponds to the rotational velocity of the pendulum axis, θ is a positional coordinate which defines the deflection angle of the pendulum axis from vertical,

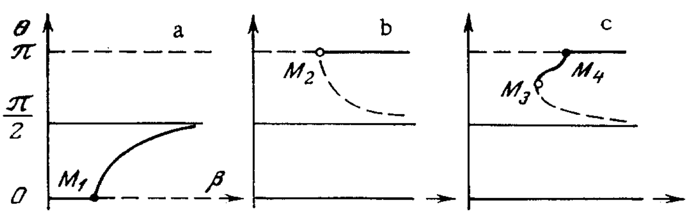

Fig. 2

and $\alpha = (B - C)/B$ (A, B, and C are, respectively, the equatorial and axial moments of inertia about the principal axes, with the coordinate origin located at the pendulum suspension point, $A = B$).

The coordinates $[\beta, \theta]$ of branching points are

$$M_1\left[\frac{(1-\alpha)^2}{\alpha}, 0\right] \quad (0 < \alpha < 1), \quad M_2\left[-\frac{(1-\alpha)^2}{\alpha}, \pi\right]\left(-\frac{1}{3} < \alpha < 0\right)$$

$$M_3\left[\frac{16}{9}\sqrt{-\frac{3}{\alpha}},\quad \arccos\left(-\sqrt{-\frac{1}{3\alpha}}\right)\right]\quad \left(\alpha<-\frac{1}{3}\right)$$

$$M_4\left[-\frac{(1-\alpha)^2}{\alpha},\ \pi\right]\quad \left(\alpha<-\frac{1}{3}\right)$$

The solid and dash lines in Fig. 2 denote, respectively, stable and unstable sections of the equilibrium curve. The nature of noncritical point stability was determined in [3] by analyzing the sign of Poincaré's stability coefficient of the considered there system. At branching points $M_1 - M_4$ the stability coefficient is zero, hence the determination of the nature of stability at these points necessitates further analysis. On the basis of theorems in Section 3 we conclude that point M_1 and M_4 are stable since both are adjoined by single stable half-branches (on the left and right, respectively); point M_3 is unstable because it is a limit point, and point M_2 is also unstable since a single unstable half-branch adjoins it on the left.

These conclusions can be directly verified by analyzing higher derivatives of the Routh potential with respect to θ at respective points.

Using the formula for the Routh potential W [3] it is possible to show that $W'''(M_3) > 0$ (primes denote differentiation with respect to θ). This proves the instability of point M_3. At points M_1, M_2, and M_4 we have $W''' = 0$ and

$$W''''(M_1) = -1 + \frac{4(1+2\alpha)}{1-\alpha},\quad W''''(M_2) = W''''(M_4) = 1 - \frac{4(1+2\alpha)}{1-\alpha}$$

where the first expression is positive when $0 < \alpha < 1$ which implies stability of point M_1. The second expression is negative when $-1/3 < \alpha < 0$ and positive when $\alpha < -1/3$, which proves the instability of point M_2 and stability of point M_4.

Note that the law of stability change is not satisfied on the vertical straight line passing through M_3 (Fig. 2, c) [1, 2] (it is of course satisfied on straight lines that do not pass through branching points).

Example 2. As an example of the case of several parameters (Sect. 4) we consider the problem of stability of the degenerate permanent rotation of a heavy symmetric body with a fixed point whose center of mass is above the suspension point. We take Euler's angles θ, ψ, and φ as the generalized coordinates and the generalized momenta β_2 and β_3 of the cyclic coordinates ψ and φ as parameters of steady motion surfaces [3, 16]. In the notation used in [16] that surface in space $(\theta, \beta_2, \beta_3)$ is defined by

$$\frac{(\beta_3 - \beta_2\cos\theta)(\beta_2 - \beta_3\cos\theta)}{A\sin^3\theta} - Mgz_0\sin\theta = 0 \tag{5.1}$$

Permanent rotations lie on the straight line $\beta_2 = \beta_3 = \beta,\ \theta = 0$, while the degenerate permanent rotation

$$\beta^2 = 4AMgz_0,\quad \theta = 0 \tag{5.2}$$

(on which the second differential of the Routh potential is degenerate) is situated at the boundary of stability region $\beta^2 - 4AMgz_0 > 0$ of that straight line [16]. It was shown in [17] that it is stable.

We shall prove the stability of that motion without analyzing higher derivatives of the Routh potential. Theorem 4.1 is applicable when at least one cross section of surface (5.1) contains a single half-branch adjoining point (5.2). Let us consider the cylindrical cross section produced by the plane $\beta_2 = \beta_3 = \beta$ for which formula (5.1) reduces to the form

$$\left(\beta^2 - 4AMgz_0\cos^4\frac{\theta}{2}\right)\sin\frac{\theta}{2} = 0 \tag{5.3}$$

At that cross section (5.2) defines a branching point. Formula (5.3) implies the existence of the single branch $\theta = 0$ when $\beta^2 - 4AMgz_0 > 0$. Since that branch is stable, point (5.2) is by Theorem 4.1 also stable, Q.E.D.

Note that when $\beta^2 - 4AMgz_0 > 0$ the stability of nondegenerate permanent rotations, as well as that of regular precessions can also be determined by the form of the stationary motion surface using the Poincaré — Chetaev theory of equilibrium bifurcation [1,2] and on the strength of the investigation in [3]. Since the instability of point $\theta = \beta_2 = \beta_3 = 0$ is evident on physical grounds, hence along the trivial branch $\theta = 0$ of cross section (5.3) the region to the left of point (5.2) is unstable, while to the right of it it is stable. That cross section nontrivial branches are directed to the left of that point and consequently, by the law of stability change [1 —3] they are stable (cf. Fig.1,b). This implies the stability of nontrivial branches of surface (5.1), since they do not contain branching points.

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